AP® CALCULUS AB/CALCULUS BC
2015 SCORING GUIDELINES

Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

(c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t)\,dt = 76.570$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$

Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

$2 : \begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t, $0 \le t \le 8$, is

$30 + \int_0^t [R(x) - D(x)]\,dx.$

$R(t) - D(t) = 0 \Rightarrow t = 0,\ 3.271658$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

$3 : \begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $30 + \int_0^w [R(t) - D(t)]\,dt = 50$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially ~~blocked, allowing water to drain~~ out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

$$\int_0^8 R(t)\,dt = \int_0^8 20\sin\frac{t^2}{35}\,dt = \boxed{76.570 \text{ ft}^3}$$

Durring the eight hour intervol, about 76.570 cubic feet of water flow into the drainpipe

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

Total water: $T(x)$ $\quad T'(x) = R(x) - D(x) = 20\sin\frac{t^2}{35} + .04t^3 - .4t^2 - .96t$

$T'(3) = 20\sin 9/35 + .04(27) - .4(9) - .96(3) = -.314 < 0$

after three hours, the amount of water in the pipe is $\boxed{\text{decreasing}}$ because the derivative of the amount of water (the difference between water entering and leaving) is less than zero at 3 hours.

Continue problem 1 on page 5.

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(c) At what time t, $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.

$T'(t) = 0$ @ $t = 0, 3.2716584$

$T(t) = T(0) + \int_0^t T'(t)\,dt = 30 + \int_0^t T'(t)\,dt$

$T(0) = 30$

$T(3.272) = 27.965$

$T(8) = 48.544$

after testing all critical numbers and endpoints for their values, the amount of water in the pipe achieves a minimum value of about 27.965 after about $\boxed{3.272 \text{ hours}}$

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

$50 = T(w)$

$50 = 30 + \int_0^w T'(t)\,dt$

$\boxed{20 = \int_0^w T'(t)\,dt} = \int_0^w [R(t) - D(t)]\,dt$

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 1 & 2 1B

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

$$\int_0^8 20\sin\left(\frac{t^2}{35}\right)dt = 76.57035295 \text{ ft}^3$$

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

At $t=3$ hours, $R(t) < D(t)$ $\therefore$ The amount of water in the pipe is decreasing

$$R(3) = 20\sin\left(\frac{3^2}{35}\right) = 5.086 \text{ ft}^3/\text{hour}$$

$$D(3) = -.04(3)^3 + .4(3)^2 + 0.96(3) = 5.4 \text{ ft}^3/\text{hour}$$

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Continue problem 1 on page 5.

(c) At what time t, $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.

$R(t) = 0 = 20 \sin\left(\frac{t^2}{35}\right)$ @ $t = 0$, 10.486

+ + +

0 10.486

abs minimum @ $t = 0$

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

$50 = 30 + \int_0^w 20 \sin\left(\frac{t^2}{35}\right) dt - \int_0^w -.04t^3 + .4t^2 + .96t \, dt$

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1C

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

$(0, 30)$

$R(8) = 20\sin\left(\frac{8^2}{35}\right)$

$= 20\sin\left(\frac{64}{35}\right)$

$\boxed{R(8) = 19.3392}$

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

$R(3) = 20\sin\left(\frac{3^2}{35}\right)$

$= 20\sin\left(\frac{9}{35}\right)$

$R(3) = 5.08637$

$D(t) = -0.04(3)^3 + 0.4(3)^2 + 0.96(3)$

$D(t) = 5.4$

The amount of water in the pipe is decreasing at time $t = 3$ hours, because the rate at which the water is draining out the other end of the pipe is greater, $D(3) = 5.4$, than the rate at which the rainwater flows into the drainpipe, $R(3) = 5.08637$.

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Continue problem 1 on page 5.

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1C

(c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

$$20\sin\left(\frac{t^2}{35}\right) = -0.04t^3 + 0.4t^2 + 0.96t$$

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

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Question 1

Overview

In this problem students were given $R(t)$, the rate of flow of rainwater into a drainpipe, in cubic feet per hour, and $D(t)$, the rate of flow of water out of the pipe, in cubic feet per hour. Both $R(t)$ and $D(t)$ are defined on the time interval $0 \le t \le 8$. The amount of water in the pipe at time $t = 0$ is also given. In part (a) students needed to use the definite integral to compute the amount of rainwater that flows into the pipe during the interval $0 \le t \le 8$. Students had to set up the definite integral $\int_0^8 R(t)\,dt$ and evaluate the integral using the calculator. In part (b) students should have recognized that the rate of change of the amount of water in the pipe at time t is given by $R(t) - D(t)$. Students were expected to calculate $R(3) - D(3)$ using the calculator and find that the result is negative. Therefore, the amount of water in the pipe is decreasing at time $t = 3$. In part (c) students had to find the time t, $0 \le t \le 8$, at which the amount of water in the pipe is at a minimum. Students were expected to set up an integral expression such as $30 + \int_0^t [R(x) - D(x)]\,dx$ for the amount of water in the pipe at time t. Students should have realized that an absolute minimum exists since they are working with a continuous function on a closed interval, and this minimum must occur at either a critical point or at an endpoint of the interval. Students were expected to use the calculator to solve $R(t) - D(t) = 0$ and find the single critical point at $t = 3.272$ on the interval $0 < t < 8$. Students should have stored the full value for t in the calculator and used the calculator to evaluate the function at the critical point and the endpoints. In this case the amount of water is at a minimum at the single critical point. In part (d) students were asked to write an equation involving one or more integrals that gives the time w when the pipe will begin to overflow. Students were expected to set up an equation using the initial condition, an integral expression, and the holding capacity of the pipe, such as

$30 + \int_0^w [R(t) - D(t)]\,dt = 50.$

Sample: 1A
Score: 9

The response earned all 9 points.

Sample: 1B
Score: 6

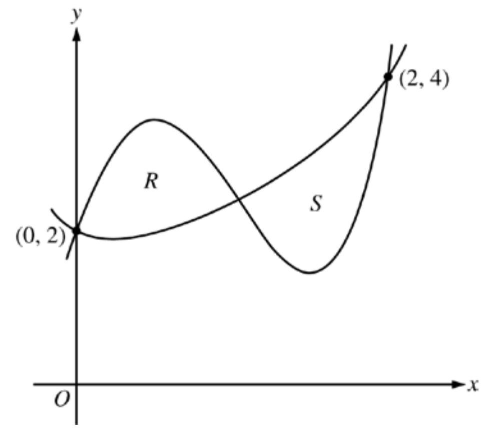
The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In parts (a) and (b), the student's work is correct. In part (c) the student works with $R(t)$ rather than $R(t) - D(t)$. In part (d) the student's work is correct.

Sample: 1C
Score: 3

The response earned 3 points: no points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student finds the rate at which water enters the pipe rather than the total amount. In part (b) the student's work is correct. In part (c) the student earned the first point for considering $R(t) - D(t) = 0$.

Question 2

Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2 - 2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.



(a) Find the sum of the areas of regions R and S.

(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

(a) The graphs of $y = f(x)$ and $y = g(x)$ intersect in the first quadrant at the points $(0, 2)$, $(2, 4)$, and $(A, B) = (1.032832, 2.401108)$.

$$\text{Area} = \int_0^A [g(x) - f(x)]\,dx + \int_A^2 [f(x) - g(x)]\,dx$$
$$= 0.997427 + 1.006919 = 2.004$$

$$4 : \begin{cases} 1 : \text{limits} \\ 2 : \text{integrands} \\ 1 : \text{answer} \end{cases}$$

(b) $\text{Volume} = \int_A^2 [f(x) - g(x)]^2\,dx = 1.283$

$$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(c) $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x)$

$h'(1.8) = f'(1.8) - g'(1.8) = -3.812$ (or -3.811)

$$2 : \begin{cases} 1 : \text{considers } h' \\ 1 : \text{answer} \end{cases}$$

 2A1

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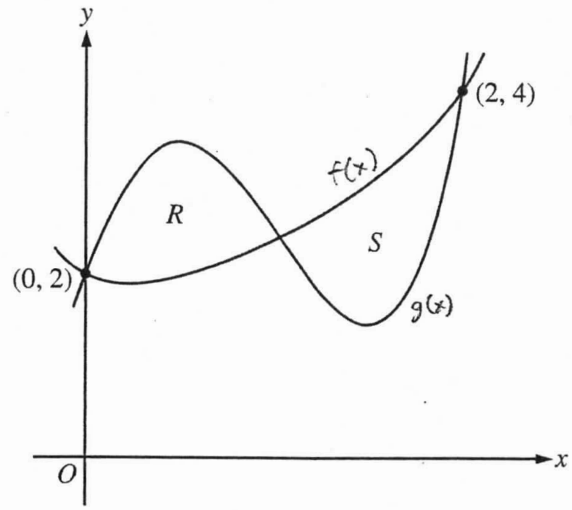

2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2 - 2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

$f(a) = g(a)$

$a = 1.0328318883641$

$$\int_0^a [g(x) - f(x)]\,dx + \int_a^2 [f(x) - g(x)]\,dx = 2.004$$

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Continue problem 2 on page 7.

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(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

vertical

Cross section area $A(x) = (f(x) - g(x))^2$ $\quad a = 1.0328318883641$

$$\int_a^2 A(x)\,dx = 1.283$$

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

$f'(1.8) = 2.1162821217136$

$g'(1.8) = 5.928$

$$\frac{dh}{dx} = f'(1.8) - g'(1.8) = -3.812$$

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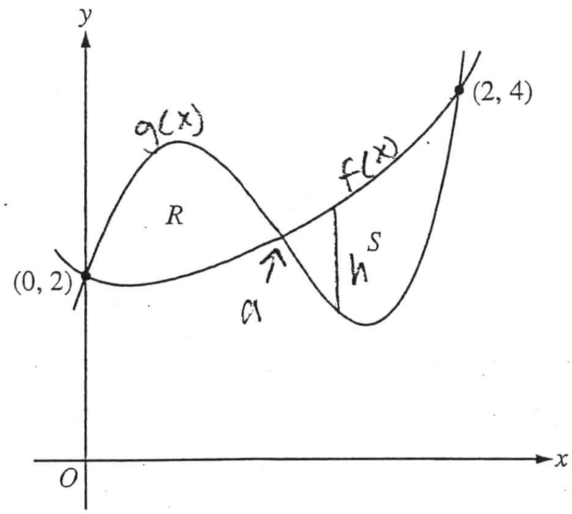

2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2-2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

$a = 1.0328319$

$R = \int_0^a (g(x) - f(x))dx = 0.997 \text{ un}^2$

$S = \int_a^2 (f(x) - g(x))dx = 1.007 \text{ un}^2$

Continue problem 2 on page 7.

(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

$$\text{Volume} = \int_a^2 ((f(x) - g(x))^2)\,dx = 1.283 \text{ un}^3$$

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

$$h = \int_a^2 (f(x) - g(x))\,dx$$

$$\frac{dh}{dt} = \int_a^2 (f'(x) - g'(x))\,dx$$

$$\frac{dh}{dt} = f(1.8) - g(1.8) = 20.449$$

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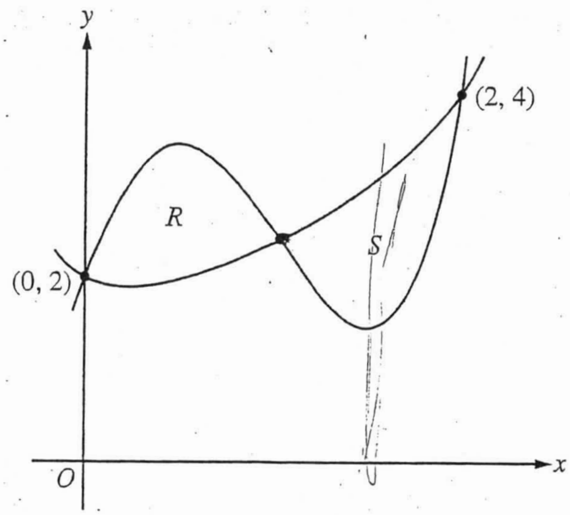

2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2-2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

$$\int_0^1 \left(\left(x^4 - 6.5x^2 + 6x + 2\right) - \left(1 + x + e^{x^2-2x}\right)\right) dx$$

$$\approx 1$$

$$\int_1^2 \left(\left(1 + x + e^{x^2-2x}\right) - \left(x^4 - 6.5x^2 + 6x + 2\right)\right) dx$$

$$\approx 1$$

$$= \boxed{2}$$

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Continue problem 2 on page 7.

(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

$$\pi\int_1^2 \left(\left(\left(1+x+e^{x^2-2x}\right)^2-\left(x^4-6.5x^2+6x+\right)^2\right)\right)$$

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

$h = f - g$

$h' = f' - g'$

$h'(1.8) = x + e^{x^2-2x} - 4x^3 - 13x + 6$

$h'(1.8) = -38.23$

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Question 2

Overview

In this problem students were given a graph of the boundary curves of two planar regions R and S in the first quadrant. One boundary curve is defined by $f(x) = 1 + x + e^{x^2 - 2x}$, and the other boundary is defined by $g(x) = x^4 - 6.5x^2 + 6x + 2$. In part (a) students were asked to find the sum of the areas of regions R and S. Two intersection points of the boundary curves, $(0, 2)$ and $(2, 4)$, are given, and students were expected to find the other point of intersection by using the calculator. The intersection point is $(A, B) = (1.032832, 2.401108)$. The sum of the areas of R and S is $\int_0^A (g(x) - f(x))\,dx + \int_A^2 (f(x) - g(x))\,dx$. Students were expected to use the calculator to evaluate the integrals. In part (b) students were asked to find the volume of a solid with S as its base. Students had to interpret the area of the cross sections as $[f(x) - g(x)]^2$, and use the calculator to evaluate the volume as $\int_A^2 [f(x) - g(x)]^2\,dx$. In part (c) students had to find the rate of change of the vertical distance, h, between the graphs of f and g at $x = 1.8$. Students were expected to recognize and communicate $h'(x) = f'(x) - g'(x)$, then evaluate $h'(1.8)$ using the numerical derivative at a point capability of the calculator.

Sample: 2A
Score: 9

The response earned all 9 points.

Sample: 2B
Score: 6

The response earned 6 points: 3 points in part (a), 3 points in part (b), and no points in part (c). In part (a) the student presents correct integrals for the areas of the two regions and earned the first 3 points. The student evaluates the areas of the two regions correctly. The student does not find the sum and did not earn the answer point. In part (b) the student's work is correct. In part (c) the student presents an incorrect expression for h'.

Sample: 2C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), and 1 point in part (c). In part (a) the student uses $x = 1$ as the x-coordinate of the point of intersection. The student did not earn the first point. For each of the regions, the student presents the correct integrand, so the second and third points were earned. The student is not eligible for the answer point. In part (b) the student presents an incorrect integrand. In part (c) the student considers $h' = f' - g'$ and earned the first point. The evaluation of $h'(1.8)$ is incorrect.

Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\,dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\,dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

(a) $v'(16) \approx \dfrac{240 - 200}{20 - 12} = 5 \text{ meters/min}^2$

1 : approximation

(b) $\int_0^{40} |v(t)|\,dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \le t \le 40$ minutes.

$$\begin{aligned}\int_0^{40} |v(t)|\,dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters}\end{aligned}$$

3 : { 1 : explanation; 1 : right Riemann sum; 1 : approximation }

(c) Bob's acceleration is $B'(t) = 3t^2 - 12t$.

$B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$

2 : { 1 : uses $B'(t)$; 1 : answer }

(d)
$$\begin{aligned}\text{Avg vel} &= \frac{1}{10}\int_0^{10} \left(t^3 - 6t^2 + 300\right) dt \\ &= \frac{1}{10}\left[\frac{t^4}{4} - 2t^3 + 300t\right]_0^{10} \\ &= \frac{1}{10}\left[\frac{10000}{4} - 2000 + 3000\right] = 350 \text{ meters/min}\end{aligned}$$

3 : { 1 : integral; 1 : antiderivative; 1 : answer }

1 of 2
3A

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

a) $v'(16) \approx \dfrac{v(20) - v(12)}{20 - 12} = \dfrac{240 - 200}{8} = \dfrac{40}{8} = \dfrac{20}{4} = 5 \dfrac{m}{min^2}$

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

b) $\int_0^{40} |v(t)| dt$ represents the total distance in meters Johanna traveled between times $t=0$ and $t=40$ minutes.

$\int_0^{40} |v(t)| dt \approx [12(200) + 8(240) + 4(220) + 16(150)]$

$= 2400 + 1920 + 880 + 2400$

$= 7600$ meters

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Continue problem 3 on page 15.

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3A 2

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

c) $\text{acceleration} = B'(t) = 3t^2 - 12t$

$$B'(5) = 3(5)^2 - 12(5)$$
$$= 75 - 60 = 15 \frac{m}{min^2}$$

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

d) $\text{Avg. velocity} = \frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300)\,dt$

$$= \frac{1}{10} \cdot \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10}$$
$$= \frac{1}{10} \cdot \left[\frac{10000}{4} - 2000 + 3000 \right]$$
$$= \frac{1}{10} [3500] = 350 \frac{m}{min}$$

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t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

$$v'(16) = \frac{v(20) - v(12)}{20 - 12} = \frac{40}{8} = 5$$

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(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

$\int_0^{40} |v(t)|\, dt$ is the total distance covered by Johanna from $t = 0$ to $t = 40$.

$200(12) + 240(8) + 220(4) + 150(16)$

$= 2400 + 1920 + 880 + 2400$

$= 4800 + 2800 = 7600$

Continue problem 3 on page 15.

NO CALCULATOR ALLOWED

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

$B'(t) = 3t^2 - 12t$

$B'(5) = 3 \cdot 25 - 12 \cdot 5$

$= 75 - 60 = 15$ (m/minute)/minute

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

$\frac{1}{10}\int_0^{10} t^3 - 6t^2 + 300 \, dt$ meters/minute

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

1 of 2

3C1

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

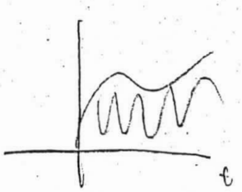
3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

$$v'(16) \approx \frac{240-200}{20-12} = \frac{40}{6} = \boxed{\frac{20}{3}}$$

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

$\int_0^{40} |v(t)|$ is the total distance travelled (including when Johanna jogs backwards as positive distance)



positive

total distance $= (12)(200) + 8(240) + 4(+220) + 6(150)$

Continue problem 3 on page 15.

NO CALCULATOR ALLOWED

3C2

(c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

$$a = \frac{dv}{dt} = 3t^2 - 12t$$

$$a(5) = 3(25) - 12(5)$$

$$75 - 60$$

$$= \boxed{15 \text{ meters/minute}^2}$$

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

$a = \frac{dv}{dt}$

$v = \int a \, dt$

$$\text{average velocity} = \frac{\int_0^{10} 3t^2 - 12t}{10 - 0}$$

$$= \left[\frac{t^3 - 6t^2}{10}\right]_0^{10} = \frac{10^3 - 6(10^2)}{10} - 0$$

$$= \frac{1000 - 600}{10}$$

$$= \frac{400}{10} = \boxed{40 \text{ meters/minute}}$$

GO ON TO THE NEXT PAGE.

Question 3

Overview

In this problem students were given a table of values of a differentiable function v, the velocity of a jogger, in meters per minute, jogging along a straight path for selected values of t in the interval $0 \le t \le 40$. In part (a) students were expected to know that $v'(16)$ can be estimated by the difference quotient $\frac{v(20) - v(12)}{20 - 12}$. In part (b) students were expected to explain that the definite integral $\int_0^{40} |v(t)|\, dt$ gives the total distance jogged, in meters, by Johanna over the time interval $0 \le t \le 40$. Students had to approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the subintervals $[0, 12]$, $[12, 20]$, $[20, 24]$, $[24, 40]$, and values from the table. In part (c) students were given a cubic function B, the velocity of a bicyclist, in meters per minute, riding along the same straight path used by Johanna for $0 \le t \le 10$. Students should have known that $B'(t)$ gives Bob's acceleration at time t. Students were expected to find $B'(t)$ using derivatives of basic functions and then evaluate $B'(5)$. In part (d) students had to set up the definite integral $\frac{1}{10}\int_0^{10} B(t)\, dt$ that gives Bob's average velocity during the interval $0 \le t \le 10$. Students needed to evaluate this integral using basic antidifferentiation and the Fundamental Theorem of Calculus.

Sample: 3A
Score: 9

The response earned all 9 points.

Sample: 3B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student's work is correct. In part (b) the student does not include "meters," so the explanation point was not earned. The student's Riemann sum and approximation are correct. In part (c) the student's work is correct. In part (d) the student's integral is correct.

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In part (a) the student attempts to simplify the correct difference quotient but makes an arithmetic error. In part (b) the student did not earn the explanation point because the time interval and the distance units (meters) are not included. The right Riemann sum has exactly one error. The student earned the point because 7 out of the 8 components are correct. The student did not earn the approximation point as a result of an error in the Riemann sum. In part (c) the student's work is correct. In part (d) the student uses $B'(t)$ in the integral instead of $B(t)$.

Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

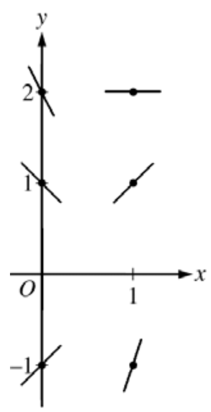
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

(a)



$$2 : \begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

$$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$$

(c) $\left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

$$2 : \begin{cases} 1 : \text{considers } \left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} \\ 1 : \text{conclusion with justification} \end{cases}$$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$2x - y = m$

$2x - (mx + b) = m$

$(2 - m)x - (m + b) = 0$

$2 - m = 0 \Rightarrow m = 2$

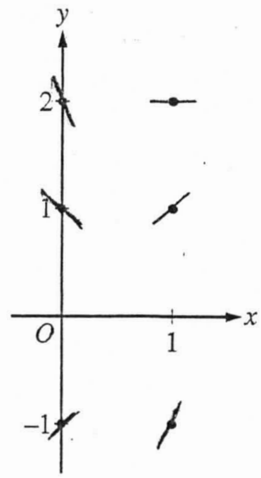
$b = -m \Rightarrow b = -2$

Therefore, $m = 2$ and $b = -2$.

$$3 : \begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$$

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$$\frac{dy}{dx} = 2x - y$$

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y)$$

$$\boxed{\frac{d^2y}{dx^2} = 2 - 2x + y}$$

In Quadrant II, $x < 0$ and $y > 0$,

so $\frac{d^2y}{dx^2} = 2 - 2x + y > 0$,

Thus $\boxed{\text{all solution curves in Quadrant II are concave up.}}$

Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$$\frac{dy}{dx} = 2x - y = 2 \cdot 2 - 3 = 1$$

$\boxed{\text{Neither}}$, as $\frac{dy}{dx} \neq 0$ at $x = 2$.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y, \quad y = mx + b$

$\frac{dy}{dx} = m = 2x - y$

$m = 2x - (mx + b)$

$m = (2-m)x - b$, equate coefficients

$2 - m = 0$

$m = 2$

$-b = m$

$b = -m = -2$

$\boxed{m = 2, b = -2}$

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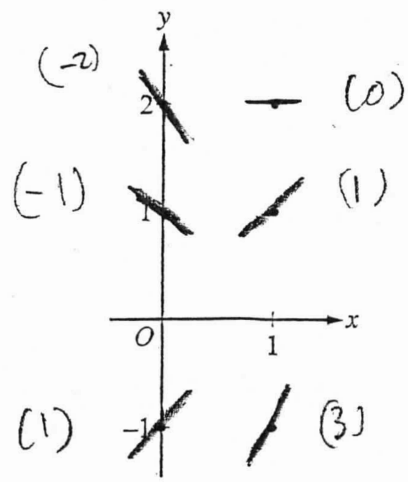
4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2 - 2x + y$$

-+ | ++

-- | +-

~~B~~ always negative

always positive

$$= 2 - 2(X) + y$$

$$= 2 + 2|x| + y = > 0$$

∴ concavity in Quadrant II is always concave up

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Continue problem 4 on page 17.

4B2
2 of 2

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$\frac{d^2y}{dx^2} = 2-2(2)+3 = 2-4+3 = -2+3 = 1 > 0$

concave up means minimum

According to the second derivative test, at $x=2$ the second derivative is positive and $\therefore$ there is a minimum at that point

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$m = 2x - y$

$y = (2x-y)x + b$

$y = 2x^2 - yx + b$

$y = 2x - \frac{dy}{dx}$

GO ON TO THE NEXT PAGE.

4C1

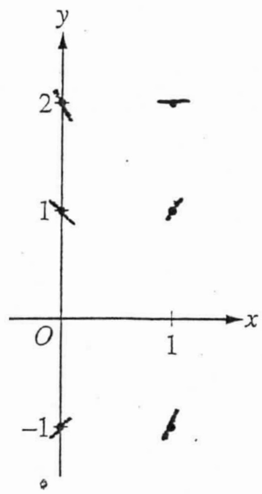
1 of 2

NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

$y \backslash x$	0	1
-1	1	3
1	-1	1
2	-2	0



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$\frac{dy}{dx} = 2x - y =$

$\frac{d^2y}{dx^2} = 2 - y'$

$0 = 2 - y'$

$0 = 2 - (2x - y)$

$0 = 2 - 2x + y$

$y = 2x - 2$

$x = \frac{y+2}{2}$

$2x - 2 = 0$

$2x = 2$

$x = 1$

$x = \frac{y+2}{2}$

$0 = y + 2$

$y = 2$

concave down at $(1,2)$ because its a relative max.

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Continue problem 4 on page 17.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$\frac{dy}{dx} = 2x - y$

$\int dy + y = \int 2x\, dx$

$y + y' = x^2$

$y' = x^2 - y$

$y' = (2)^2 - (3)$

$y' = 1$

f has a relative min at $x = 2$ because at $f(2)$ it is 3.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y$

$\int dy + y = \int 2x\, dx$

$y + y' = x^2 + c$

$y' = x^2 - y + c$

$y' = (2)^2 - 3 + c$

$y' = c$

$y - 3 = m(x - 2)$

$y = x + 1$

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Question 4

Overview

In this problem students were to consider the first-order differential equation $\frac{dy}{dx} = 2x - y$. In part (a) students were given an xy-plane with 6 labeled points and were expected to sketch a slope field by drawing a short line segment at each of the six points with slopes of $2x - y$. In part (b) students needed to use implicit differentiation and the fact that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ to obtain $\frac{d^2y}{dx^2} = 2 - 2x + y$. Students were expected to explain that for points in Quadrant II, $x < 0$ and $y > 0$ so $\frac{d^2y}{dx^2} > 0$. Thus, any solution curve for the differential equation that passes through a point (x, y) in Quadrant II must be concave up at (x, y). In part (c) students were asked to consider the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 3$. Students had to determine if $(2, 3)$ is the location of a relative minimum, a relative maximum, or neither for f and justify the answer. Students were expected to show that $\frac{dy}{dx} \neq 0$ at $(2, 3)$ and conclude that $(2, 3)$ is neither the location of a relative minimum nor a relative maximum. In part (d) students were asked to find the values of the constants m and b so that the linear function $y = mx + b$ satisfies the differential equation $\frac{dy}{dx} = 2x - y$. Students were expected to show that if $y = mx + b$, then $\frac{dy}{dx} = m$. Using a substitution in $\frac{dy}{dx} = 2x - y$ leads to $2x - y = m$ and thus $2x - (mx + b) = m$. This equation enabled the student to find the values of m and b.

Sample: 4A
Score: 9

The response earned all 9 points.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student's work is correct. In part (b) the student's work is correct, including correct reasoning about the sign of the second derivative using the signs of x and y in Quadrant II. In part (c) the student does not consider $\frac{dy}{dx}$, so the first point was not earned. The student considers $\frac{d^2y}{dx^2}$, which cannot be used as justification, and the student incorrectly identifies $x = 2$ as a minimum. The second point was not earned. In part (d) the student earned the first 2 points for declaring that $m = 2x - y$. In doing so, the student communicates that the derivative of the linear function is its slope m (the first point) and connects the differential equation and its linear solution by equating the derivatives (the second point). The student does not arrive at an answer.

Sample: 4C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for the correct second derivative in x and y, shown in the work where the student writes $0 = 2 - (2x - y)$. In part (c) the student incorrectly solves

Question 4 (continued)

the differential equation and then, from that work, finds an incorrect expression for the first derivative. The student is not eligible for any points. In part (d) the student attempts to solve the differential equation by separation of variables and uses the point $(2, 3)$, which is not relevant to the question asked.

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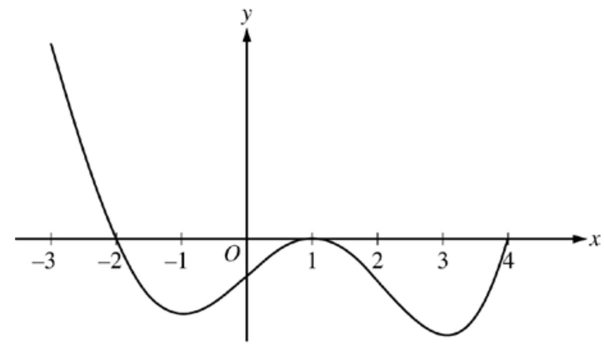
Question 5

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.



Graph of f'

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

(a) $f'(x) = 0$ at $x = -2$, $x = 1$, and $x = 4$.
$f'(x)$ changes from positive to negative at $x = -2$.
Therefore, f has a relative maximum at $x = -2$.

$2 : \begin{cases} 1 : \text{identifies } x = -2 \\ 1 : \text{answer with reason} \end{cases}$

(b) The graph of f is concave down and decreasing on the intervals $-2 < x < -1$ and $1 < x < 3$ because f' is decreasing and negative on these intervals.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(c) The graph of f has a point of inflection at $x = -1$ and $x = 3$ because f' changes from decreasing to increasing at these points.

The graph of f has a point of inflection at $x = 1$ because f' changes from increasing to decreasing at this point.

$2 : \begin{cases} 1 : \text{identifies } x = -1, 1, \text{and } 3 \\ 1 : \text{reason} \end{cases}$

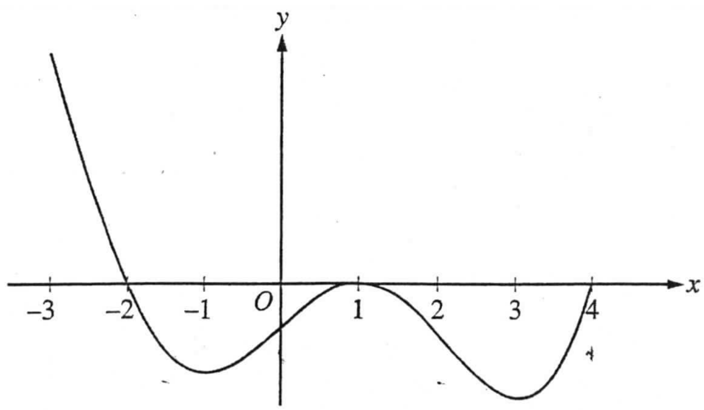
(d) $f(x) = 3 + \int_1^x f'(t)\,dt$

$f(4) = 3 + \int_1^4 f'(t)\,dt = 3 + (-12) = -9$

$$f(-2) = 3 + \int_1^{-2} f'(t)\,dt = 3 - \int_{-2}^{1} f'(t)\,dt$$
$$= 3 - (-9) = 12$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{expression for } f(x) \\ 1 : f(4) \text{ and } f(-2) \end{cases}$

1&2
5A₁



Graph of f'

5. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

$f(x)$ has a relative maximum at $x = -2$ because $f'(x)$ switches from positive to negative ~~at~~ this point.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

The graph of f is both concave down and decreasing on the intervals $(-2, -1)$ and $(1, 3)$ because on these intervals $f'(x)$ is negative and also $f''(x)$ is negative.

Continue problem 5 on page 19.

(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

$x = -1, 1, 3$

The x-coordinates of ~~the~~ the points of inflection for the graph of f are $x=-1$, $x=1$, $x=3$. This is because at these points, $f''(x)$ switches signs.

(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

$$\boxed{f(x) = \int_1^x f'(t)\,dt + 3}$$

$$f(4) = \int_1^{(4)} f'(t)\,dt + 3$$
$$= (-12) + 3$$
$$\boxed{f(4) = -9}$$

$$f(-2) = \int_1^{(-2)} f'(t)\,dt + 3$$
$$= (9) + 3$$
$$\boxed{f(-2) = 12}$$

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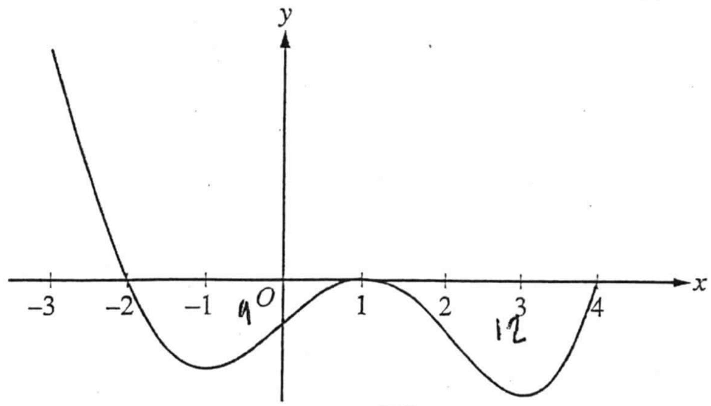
1 of 2

5B.



Graph of $\boxed{f'}$

5. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

at $x=-2$, f has a relative max because this is where $f'(x)$ changes from positive to negative.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

$(-2,-1)$ and $(1,3)$ because this is where $f'(x)$ is negative and the slope of $f'(x)$ is negative.

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Continue problem 5 on page 19.

(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

the graph of f has points of inflection at x=-1, x=1, and x=3 because this is where f'(x) has a slope of 0.

(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

$f(x) = \int f'(x)dx$

$f(4) = \int_1^4 f'(x)dx = -12$

$f(-2) = \int_{-2}^1 f'(x)dx = -9$

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1 of 2

5C1

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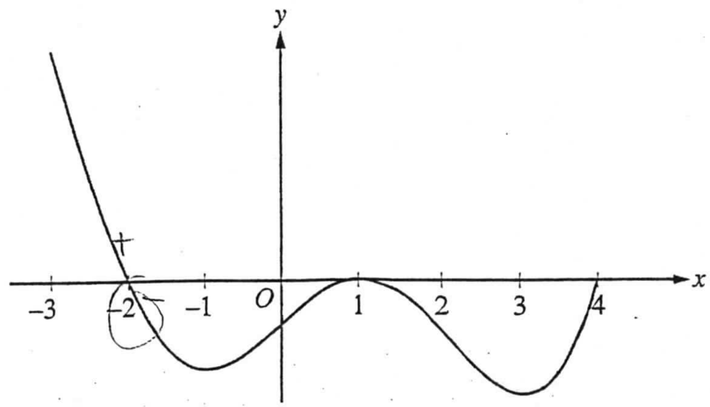

Graph of f'

5. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

f has a relative maximum at point −2 because the graph of f' at −2 goes from positive to negative.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

It is concave down at point −2 because it has a relative maximum at that point.

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Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

It has a point of inflection at point 1 because at that point it equals zero but it never passes the x-axis.

(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

$$\int_1^4 f'(x)dx = f(4) - f(1)$$

$$\int_1^4 f'(x)dx = f(4) - 3$$

$$+3 \qquad\qquad +3$$

$$f(4) = \int_1^4 f'(x)dx + 3$$

$$\int_{-2}^{1} f'(x)dx = f(1) - f(-2)$$

$$\int_{-2}^{1} f'(x)dx = 3 - f(-2)$$

$$f(-2) = 3 - \int_{-2}^{1} f'(x)dx$$

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Question 5

Overview

In this problem students were given the graph of f', the derivative of a twice-differentiable function f on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ were given. In part (a) students had to find all x-coordinates at which f has a relative maximum. Students had to find the critical points $x = -2$, $x = 1$, and $x = 4$ from the graph of f' and apply the First Derivative Test to conclude that the relative maximum occurs at $x = -2$. In part (b) students were asked to determine the open intervals where the graph of f is both concave down and decreasing. Students needed to use the graph of f' to determine the open intervals where f' was both decreasing and negative in order to answer the question. In part (c) students were asked to find the x-coordinates of all points of inflection for the graph of f. Students needed to use the graph of f' to determine the x-coordinates of the points where f' changes from increasing to decreasing or from decreasing to increasing in order to answer the question. In part (d) students were asked to write an expression for $f(x)$ that involves an integral given that $f(1) = 3$. Students were expected to use the Fundamental Theorem of Calculus to produce $f(x) = 3 + \int_1^x f'(t)\,dt$. Students had to use properties of the definite integral, including the relationship of the definite integral to the areas of the bounded regions to find $f(4)$ and $f(-2)$.

Sample: 5A
Score: 9

The response earned all 9 points.

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 1 point in part (c), and 1 point in part (d). In parts (a) and (b), the student's work is correct. In part (c) the student correctly identifies the x-coordinates of the points of inflection, so the first point was earned. The student's reason is not sufficient to earn the second point since "$f'(x)$ has a slope of 0" does not, in general, guarantee a point of inflection. In part (d) the student earned the first point for having $f'(x)$ as the integrand in a definite integral. The student does not provide an expression for $f(x)$ nor calculate correct values for $f(4)$ and $f(-2)$, so the second and third points were not earned.

Sample: 5C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), no points in part (c), and 1 point in part (d). In part (a) the student's work is correct. In part (b) the student does not identify any intervals. In part (c) the student identifies only one of the three correct values, so the first point was not earned. The student's reason is not sufficient to earn the second point. In part (d) the student earned the first point for having $f'(x)$ as the integrand in a definite integral. The student does not provide an expression for $f(x)$ nor calculate correct values for $f(4)$ and $f(-2)$, so the second and third points were not earned.

Question 6

Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

(a) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(-1,\,1)} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{4}$

An equation for the tangent line is $y = \frac{1}{4}(x+1) + 1$.

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{equation for tangent line} \end{cases}$

(b) $3y^2 - x = 0 \Rightarrow x = 3y^2$

So, $y^3 - xy = 2 \Rightarrow y^3 - (3y^2)(y) = 2 \Rightarrow y = -1$

$(-1)^3 - x(-1) = 2 \Rightarrow x = 3$

The tangent line to the curve is vertical at the point $(3, -1)$.

$3 : \begin{cases} 1 : \text{sets } 3y^2 - x = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{coordinates} \end{cases}$

(c) $\frac{d^2y}{dx^2} = \frac{(3y^2 - x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$

$$\left.\frac{d^2y}{dx^2}\right|_{(x,\,y)=(-1,\,1)} = \frac{\left(3\cdot 1^2 - (-1)\right)\cdot\frac{1}{4} - 1\cdot\left(6\cdot 1\cdot\frac{1}{4} - 1\right)}{\left(3\cdot 1^2 - (-1)\right)^2}$$

$$= \frac{1 - \frac{1}{2}}{16} = \frac{1}{32}$$

$4 : \begin{cases} 2 : \text{implicit differentiation} \\ 1 : \text{substitution for } \frac{dy}{dx} \\ 1 : \text{answer} \end{cases}$

6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

Slope $= \frac{dy}{dx}$ @ $(-1,1)$: $\frac{1}{3(1)^2-(-1)} = \frac{1}{4}$

Point $= (-1,1)$

$y - 1 = \frac{1}{4}(x+1)$

$y - 1 = \frac{x}{4} + \frac{1}{4}$

$\boxed{y = \frac{1}{4}x + \frac{5}{4}}$

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

Vertical tangent line $\Longrightarrow$ slope undefined

$\Longrightarrow$ denominator $= 0$ $\Longrightarrow$ $3y^2 - X = 0$

$3y^2 = X$

Substitute "x" for "$3y^2$" in the equation of the curve

$y^3 - [3y^2 \cdot y] = 2$

$y^3 - 3y^3 = 2$

$-2y^3 = 2$

$y^3 = -1$

$y = -1$

$(-1)^3 - x \cdot (-1) = 2$

$-1 + x = 2$

$X = 3$

$\boxed{\text{Vertical tangents at } (3,-1)}$

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(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$. Point : $(-1,1)$

$\frac{dy}{dx}$ @ $(-1,1) = \frac{1}{4}$

$3y^2 - x = 3 - (-1) = 4$

$$\frac{d^2y}{dx^2} = \frac{\left[y' \cdot (3y^2 - x)\right] - \left[(6y\overset{\frac{6}{4}}{y'} - \overset{\frac{4}{4}}{1}) \cdot (y)\right]}{(3y^2 - x)^2}$$

$$= \frac{\left(\frac{1}{\cancel{4}} \cdot \cancel{4}\right) - \left(\frac{1}{2} \cdot 1\right)}{16} = \frac{1 - \frac{1}{2}}{16} = \frac{\frac{1}{2}}{16} = \frac{1}{2} \cdot \frac{1}{16}$$

$$= \boxed{\frac{1}{32}}$$

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6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

$$\frac{dy}{dx} = \frac{1}{3 \cdot 1^2 - (-1)} = -\frac{1}{4}$$

$$\boxed{y - 1 = \frac{1}{4}(x+1)}$$

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

$3y^2 - x = 0$ ← derivative DNE when denominator $= 0$

$3y^2 - x + 2 = 2$

$y^3 - xy = 3y^2 - x + 2$ ← set equations equal to find x and y

$2 = y^3 - 3y^2 - xy + x$

$2 = y^2(y-3) - x(y+1)$

Continue problem 6 on page 21.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}(3y^2 - x) - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2-x)^2}$$

$$= \frac{\left(\frac{y}{\cancel{3y^2-x}}\right)(\cancel{3y^2-x}) - y\left(6y\left(\frac{y}{3y^2-x}\right) - 1\right)}{(3y^2-x)^2}$$

$$\frac{1 - 1\left(6\cdot 1\left(\frac{1}{3+1}\right) - 1\right)}{(3+1)^2}$$

$$\frac{1 - \frac{3}{2}\cancel{1}}{16}$$

$$-\frac{3}{2}\cdot\frac{1}{16} = \boxed{\frac{-3}{32}}$$

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NO CALCULATOR ALLOWED

6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

$m = \frac{dy}{dx} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{5}$

$y - 1 = \frac{1}{5}(x + 1)$

$\frac{1}{5}x + \frac{1}{5}$

$+1$

$\boxed{y = \frac{1}{5}x + \frac{6}{5}}$

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

$\frac{1}{5}x + \frac{6}{5} = \frac{y}{3y^2 - x}$

$(3y^2 - x)(\frac{1}{5}x + \frac{6}{5} = y$

$\frac{3}{5}yx^2 + \frac{18}{5}y^2 - \frac{1}{5}x^2 - 6/5\,x = y$

$(-1, 1)$?

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Continue problem 6 on page 21.

NO CALCULATOR ALLOWED

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

$$\frac{(y'(3y^2 - x)) - y(6yy' - 1)}{(3y^2 - x)^2} = \frac{d^2y}{dx^2}$$

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Question 6

Overview

In this problem students were given the equation of a curve, $y^3 - xy = 2$, with $\frac{dy}{dx} = \frac{y}{3y^2 - x}$. In part (a) students had to find an equation for the line tangent to the curve at the point $(-1, 1)$. Students were expected to use the given $\frac{dy}{dx}$ to find the slope of the curve at the point $(-1, 1)$. In part (b) students were asked to find the coordinates of all points on the curve at which there is a vertical tangent line. These are the points on the curve where $3y^2 - x = 0$, but $y \neq 0$. Students were expected to solve $y^3 - xy = 2$ with the condition that $3y^2 - x = 0$ and report only those pairs (x, y) where $y \neq 0$. In part (c) students were asked to evaluate $\frac{d^2y}{dx^2}$ at the point $(-1, 1)$ on the curve. Students had to use implicit differentiation with $\frac{dy}{dx}$ to find an expression for $\frac{d^2y}{dx^2}$, which required use of the chain rule and either the product rule or the quotient rule. The expression can be written in terms of x and y or can involve $\frac{dy}{dx}$. In either case, students needed to evaluate the expression for $\frac{d^2y}{dx^2}$ at $(-1, 1)$.

Sample: 6A
Score: 9

The response earned all 9 points.

Sample: 6B
Score: 6

The response earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the student's work is correct. In part (b) the student considers the equation $3y^2 - x = 0$, so the first point was earned. The student does not present an equation in one variable. In part (c) the student correctly differentiates and substitutes for $\frac{dy}{dx}$, so the first 3 points were earned. The student makes an error in computation, so the answer point was not earned.

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), and 2 points in part (c). In part (a) the student makes an arithmetic error in computing the slope, so the first point was not earned. The student uses the slope to present a line that passes through $(-1, 1)$, so the second point was earned. In part (b) the student does not consider the equation $3y^2 - x = 0$. In part (c) the student correctly differentiates $\frac{dy}{dx}$, so the first 2 points were earned.